COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

KAMRAN DIVAANI-AAZAR AND REZA SAZEEDEH

Abstract. Let $\mathfrak{a}$ denote an ideal of a commutative Noetherian ring R and M and N two finitely generated R-modules with $\operatorname{pd} M < \infty$. It is shown that if $\mathfrak{a}$ is principal or R is complete local and $\mathfrak{a}$ a prime ideal with $\dim R/\mathfrak{a} = 1$, then the generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$. This provides an affirmative answer for the above ideal $\mathfrak{a}$ to a question proposed in [**13**].

1. Introduction

A generalization of local cohomology functors has been given by J. Herzog in [**6**]. Let $\mathfrak{a}$ denote an ideal of a commutative Noetherian ring R. For each $i \geq 0$, the functor $H^i_{\mathfrak{a}}(.,.)$ defined by $H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$, for all R-modules M and N. Clearly, this notion is a generalization of the usual local cohomology functor. The study of this concept was continued in the articles [**10**], [**2**] and [**12**]. Recently, there is some new interest on studying generalized local cohomology (see e.g. [**1**], [**13**] and [**14**]).

In 1969, A. Grothendieck conjectured that if $\mathfrak{a}$ is an ideal of R and N is a finitely generated R-module, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(N))$ is finitely generated for all $i \geq 0$. R. Hartshorne provides a counter-example to this conjecture in [**5**]. He defined a module N to be $\mathfrak{a}$-cofinite if $\operatorname{Supp}_R N \subseteq \operatorname{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is finitely generated for all $i \geq 0$ and he asked the following question.

Question 1.1. Let $\mathfrak{a}$ be an ideal of R and N a finitely generated R-module. When are $H^i_{\mathfrak{a}}(N)$ $\mathfrak{a}$-cofinite for all $i \geq 0$?

Hartshorne proved that if $\mathfrak{a}$ is an ideal of the complete regular local ring R and N a finitely generated R-module, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite in two cases:
(i) (see [**5**, Corollary 6.3]) $\mathfrak{a}$ is a principal ideal, and

2000 *Mathematics Subject Classification.* 13D45, 14B15.

Key words and phrases. Generalized local cohomology, Cofiniteness, Spectral sequences.

(ii) (see [**5**, Corollary 7.7]) $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$.

This subject was studied by several authors afterward (see e.g. [**8**],[**4**] and [**15**]). The best result concerning cofiniteness of local cohomology is:

Theorem 1.2. *([**8**],[**4**] and [**15**]) Let $\mathfrak{a}$ be an ideal of R and N a finitely generated R-module. If either $\mathfrak{a}$ is principal or R is local and $\dim R/\mathfrak{a} = 1$, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$.*

S. Yassemi [**13**, Question 2.7] asked wether 1.2 holds for generalized local cohomology. The main aim of this paper is to extend 1.2 to generalized local cohomology. More precisely, we prove the following.

Theorem 1.3. *Let $\mathfrak{a}$ denote an ideal of the ring R. Let M and N be two finitely generated R-modules with $\operatorname{pd} M < \infty$. If either*

(i) *$\mathfrak{a}$ is principle, or*

(ii) *R is complete local and $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$,*

then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$.

All rings considered in this paper are assumed to be commutative Noetherian with identity. In our terminology we follow that of the text book [**3**].

2. Cofiniteness results

Let $\mathfrak{a}$ denote an ideal of a ring R. The generalized local cohomology defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N),$$

for all R-modules M and N. Note that this is in fact a generalization of the usual local cohomology, because if $M = R$, then $H^i_{\mathfrak{a}}(R, N) = H^i_{\mathfrak{a}}(N)$.

Definition 2.1. Let M be an R-module. The generalized ideal transform functor with respect to an ideal $\mathfrak{a}$ of R is defined by

$$D_{\mathfrak{a}}(M, \cdot) = \varinjlim_n \operatorname{Hom}_R(\mathfrak{a}^n M, \cdot).$$

Let $R^i D_{\mathfrak{a}}(M, \cdot)$ denote the i-th right derived functor of $D_{\mathfrak{a}}(M, \cdot)$. One can check easily that there is a natural isomorphism $R^i D_a(M, \cdot) \cong \varinjlim_n \operatorname{Ext}^i_R(\mathfrak{a}^n M, \cdot)$. Thus, by

considering the Ext long exact sequences induced by the short exact sequences

$$0 \longrightarrow \mathfrak{a}^n M \longrightarrow M \longrightarrow M/\mathfrak{a}^n M \longrightarrow 0, \quad (n \in \mathbb{N}),$$

we can deduce the following lemma.

Lemma 2.2. *Let M be an R-module. For any R-module N, there is an exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow D_{\mathfrak{a}}(M,N) \longrightarrow H^1_{\mathfrak{a}}(M,N) \longrightarrow \dots$$
$$\longrightarrow \dots \longrightarrow H^i_{\mathfrak{a}}(M,N) \longrightarrow \mathrm{Ext}^i_R(M,N) \longrightarrow R^i D_{\mathfrak{a}}(M,N) \longrightarrow H^{i+1}_{\mathfrak{a}}(M,N) \longrightarrow \dots.$$

Moreover, if M has finite projective dimension, then there is a natural isomorphism $H^{i+1}_{\mathfrak{a}}(M,N) \cong R^i D_{\mathfrak{a}}(M,N)$ for all $i \geq \mathrm{pd}\, M + 1$.

Let $\mathfrak{a}$ be an ideal of R and M an R-module. We say that M is $\mathfrak{a}$-cofinite, if $\mathrm{Supp}_R\, M \subseteq V(\mathfrak{a})$ and $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$.

Lemma 2.3. *Suppose M, N are two R-modules and $\mathfrak{a}$ an ideal of R. If M is finitely generated, then $\mathrm{Supp}_R\, H^i_{\mathfrak{a}}(M,N) \subseteq V(\mathfrak{a})$, for all $i \geq 0$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. It follows from [**9**, Theorem 9.50], that

$$\mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)_{\mathfrak{p}} \cong \mathrm{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/(\mathfrak{a}^n R_{\mathfrak{p}})M_{\mathfrak{p}}, N_{\mathfrak{p}}),$$

for all $i \geq 0$. On the other hand, it is well known that the formation tensor product preserves direct limits (see e.g. [**9**, Corollary 2.20]). Thus

$$H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}} \cong R_{\mathfrak{p}} \otimes_R \varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N) \cong \varinjlim_n \mathrm{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/(\mathfrak{a}^n R_{\mathfrak{p}})M_{\mathfrak{p}}, N_{\mathfrak{p}}) \cong H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}).$$

This shows that $\mathrm{Supp}_R\, H^i_{\mathfrak{a}}(M,N) \subseteq V(\mathfrak{a})$, as required. ■

Lemma 2.4. (i) *If M is a finitely generated R-module such that $\mathrm{Supp}_R\, M \subseteq V(\mathfrak{a})$, then M is $\mathfrak{a}$-cofinite.*

(ii) *Let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequences of R-modules. Whenever two of L, M or N are $\mathfrak{a}$-cofinite, then the third one is also $\mathfrak{a}$-cofinite.*

Proof. (i) Since M is finitely generated, it follows that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$, is finitely generated for all $i \geq 0$. Hence M is $\mathfrak{a}$-cofinite, by definition.

(ii) This is well known and can be deduced easily, by considering the long exact sequence

$$\dots \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}, L) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \longrightarrow \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, L) \longrightarrow \dots.$$

■

Lemma 2.5. *Let $\mathfrak{a} = Ra$ be a principal ideal of R and M and N two finitely generated R-modules. Let $\operatorname{Hom}_R(M,N)_a$ denote the localization of $\operatorname{Hom}_R(M,N)$ with respect to the multiplicative closed subset $\{a^i : i \geq 0\}$ of R. Then*

(i) *there is a natural isomorphism $D_{\mathfrak{a}}(M,N) \cong \operatorname{Hom}_R(M,N)_a$, and*

(ii) *$H^1_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite.*

Proof. (i) If a is nilpotent, then it is clear that both $D_{\mathfrak{a}}(M,N)$ and $\operatorname{Hom}_R(M,N)_a$ will vanish. Hence, we may and do assume that a is not nilpotent. For all $i, j \in \mathbb{N}$ with $j \geq i$, let $\pi_{ij} : \operatorname{Hom}_R(a^iM, N) \longrightarrow \operatorname{Hom}_R(a^jM, N)$ be the map defined by $\pi_{ij}(f) = f|_{a^jM}$, for all $f \in \operatorname{Hom}_R(a^iM, N)$. Also, denote the natural map $\operatorname{Hom}_R(a^iM, N) \longrightarrow D_{\mathfrak{a}}(M,N)$, by π_i. Recall that, we defined $D_{\mathfrak{a}}(M,N)$ as the direct limit of the direct system $(\operatorname{Hom}_R(a^iM,N), \pi_{ij})_{i,j\in\mathbb{N}}$.

Now define $\psi_i : \operatorname{Hom}_R(a^iM, N) \longrightarrow (\operatorname{Hom}_R(M,N))_a$, by $\psi_i(f) = f\lambda_i/a^i$, where $\lambda_i : M \longrightarrow a^iM$ is defined by $\lambda_i(m) = a^im$, for all $m \in M$. Clearly $\{\psi_i\}_{i\in\mathbb{N}}$ is a morphism between direct systems. Assume $\psi : D_{\mathfrak{a}}(M,N) \longrightarrow (\operatorname{Hom}_R(M,N))_a$ is the homomorphism induced by $\{\psi_i\}_{i\in\mathbb{N}}$. Thus for each $g \in D_{\mathfrak{a}}(M,N)$, we have $\psi(g) = \psi_i(f)$, where $i \in \mathbb{N}$ and $f \in \operatorname{Hom}_R(a^iM,N)$ are such that $\pi_i(f) = g$. We show that ψ is an isomorphism. First, we show that ψ is injective. Suppose $\psi(g) = 0$, for some $g \in D_{\mathfrak{a}}(M,N)$. There are $i \in \mathbb{N}$ and $f \in \operatorname{Hom}_R(a^iM,N)$, such that $g = \pi_i(f)$. Hence

$$\psi(g) = \psi_i(f) = f\lambda_i/a^i = 0.$$

Hence there is $t \in \mathbb{N}$ such that $a^t(f\lambda_i) = 0$. Set $j = i + t$. Then it follows that $\pi_{ij}(f) = 0$ and so

$$g = \pi_i(f) = \pi_j(\pi_{ij}(f)) = 0.$$

Next, we show that ψ is surjective. Let $x_1, x_2, \dots, x_t$ be a set of generators of M. Let $l \in (\operatorname{Hom}_R(M,N))_a$. Then there are $h \in \operatorname{Hom}_R(M,N)$ and $c \in \mathbb{N}$, such that $l = h/a^c$. Since N is a Noetherian R-module, there exists an integer $e \geq c$, such that $(0 :_N a^e) = (0 :_N a^{e+j})$, for all $j \geq 0$. Define $f \in \operatorname{Hom}_R(a^{2e}M, N)$, by $f(a^{2e}x) = a^{2e-c}h(x)$, for all $x \in M$. If $a^{2e}x = a^{2e}x'$, for some x and x' in M, then

$h(x-x') \in (0:_N a^{2e})$. Hence $a^{2e-c}h(x) = a^{2e-c}h(x')$. Therefore f is well-defined. Set $g = \pi_{2e}(f)$. Then

$$\psi(g) = \psi_{2e}(f) = f\lambda_{2e}/a^{2e} = h/a^c = l.$$

Thus ψ is surjective.

(ii) Let $\psi : D_{\mathfrak{a}}(M,N) \longrightarrow (\mathrm{Hom}_R(M,N))_a$ be as above. By part (i), [**3**, Theorem 2.2.4(i)] and 2.2 we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0_{\mathfrak{a}}(M,N) & \longrightarrow \mathrm{Hom}_R(M,N) & \xrightarrow{f} & D_a(M,N) & \longrightarrow & H^1_{\mathfrak{a}}(M,N) & \xrightarrow{g} & \mathrm{Ext}^1_R(M,N) \\
 & & \Big\downarrow id & & \Big\downarrow \psi & & & & \\
0 \longrightarrow & \Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,N)) & \longrightarrow \mathrm{Hom}_R(M,N) & \xrightarrow{h} & \mathrm{Hom}_R(M,N)_a & \longrightarrow & H^1_{\mathfrak{a}}(\mathrm{Hom}_R(M,N)) & \longrightarrow 0. &
\end{array}$$

Let K be the kernel of the map g. We have $K \cong \operatorname{coker} f$ and $H^1_{\mathfrak{a}}(\mathrm{Hom}_R(M,N)) \cong \operatorname{coker} h$. The map ψ induces an isomorphism $\psi^* : \operatorname{coker} f \longrightarrow \operatorname{coker} h$, which is defined by $\psi^*(x + imf) = \psi(x) + imh$, for all $x + imf \in \operatorname{coker} f$. Hence, it follows that $K \cong H^1_{\mathfrak{a}}(\mathrm{Hom}_R(M,N))$. Therefore K is $\mathfrak{a}$-cofinite, by 1.2. Now consider the exact sequence

$$0 \longrightarrow K \longrightarrow H^1_{\mathfrak{a}}(M,N) \longrightarrow \operatorname{im} g \longrightarrow 0.$$

Since $\mathrm{Ext}^1_R(M,N)$ is finitely generated, it follows by, 2.3 and 2.4(i), that $\operatorname{im} g$ is $\mathfrak{a}$-cofinite. Thus $H^1_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite, by 2.4(ii). ■

Lemma 2.6. *Let $\mathfrak{a}$ denote an ideal of the ring R and N an $\mathfrak{a}$-cofinite R-module. Suppose that for any finitely generated R-module M with $\operatorname{pd} M < \infty$, $\mathrm{Hom}_R(M,N)$ (resp. $M \otimes_R N$) is $\mathfrak{a}$-cofinite. Then $\mathrm{Ext}^i_R(M,N)$ (resp. $\mathrm{Tor}^R_i(M,N)$) is $\mathfrak{a}$-cofinite for all finitely generated R-modules M with $\operatorname{pd} M < \infty$ and all $i \geq 0$.*

Proof. We prove only the $\mathfrak{a}$-cofiniteness of $\mathrm{Ext}^i_R(M,N)$, $i \geq 0$ and the proof of the other part is similar. The proof proceeds by induction on $t = \operatorname{pd} M$. For $t = 0$, the claim holds by assumption. Now, suppose $t > 0$. There is a short exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

From this sequence, we deduce the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(R^n,N) \longrightarrow \mathrm{Hom}_R(K,N) \longrightarrow \mathrm{Ext}^1_R(M,N) \longrightarrow 0,$$

and the isomorphisms $\mathrm{Ext}^{i+1}_R(M,N) \cong \mathrm{Ext}^i_R(K,N)$ for all $i \geq 1$. Thus from induction hypothesis, we deduce that $\mathrm{Ext}^{i+1}_R(M,N)$ is $\mathfrak{a}$-cofinite for all $i \geq 1$. Note that $\operatorname{pd} K < t$. Also, by using the above exact sequence, one can check easily that $\mathrm{Ext}^1_R(M,N)$ is $\mathfrak{a}$-cofinite. Therefore, the claim follows by induction. ■

Lemma 2.7. *Let $\mathfrak{a}$ denote an ideal of the ring R. Let M and N be two finitely generated R-modules with $\operatorname{pd} M < \infty$. If either*

(i) *$\mathfrak{a}$ is principal, or*

(ii) *R is complete local and $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$,*

then $\operatorname{Ext}^p_R(M, H^q_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-cofinite for all $p, q \geq 0$

Proof. First, we consider the case that $\mathfrak{a}$ is principal. By [**9**, Theorem 11.38], there is a Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M, N).$$

We have $E_2^{p,q} = 0$ for $q \neq 0, 1$, because, by [**3**, Theorem 3.3.1], $H^q_{\mathfrak{a}}(N) = 0$, for all $q > 1$. Since $E_2^{p,0}$ is finitely generated and $\operatorname{Supp}_R E_2^{p,0} \subseteq V(\mathfrak{a})$, it follows by 2.4(i), that $E_2^{p,0}$ is $\mathfrak{a}$-cofinite. Therefore, it is enough to show that $E_2^{p,1}$ is $\mathfrak{a}$-cofinite for all $p \geq 0$. By [**9**, Corollary 11.44], we have an exact sequence

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1_{\mathfrak{a}}(M, N) \xrightarrow{g} E_2^{0,1} \longrightarrow E_2^{2,0} \xrightarrow{f} H^2_{\mathfrak{a}}(M, N).$$

By 2.5(ii), $H^1_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite. Thus, it turns out that $\operatorname{im} g$ is $\mathfrak{a}$-cofinite, by 2.4(ii). From the exact sequence

$$0 \longrightarrow \operatorname{im} g \longrightarrow E_2^{0,1} \longrightarrow \ker f \longrightarrow 0,$$

we deduce that $E_2^{0,1}$ is $\mathfrak{a}$-cofinite. Note that $\ker f$ is a finitely generated R-module. Therefore 2.6 implies that $E_2^{p,1}$ is $\mathfrak{a}$-cofinite for all $p \geq 0$, because $H^1_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite by 1.2.

Now suppose that R is a complete local ring and $\mathfrak{a}$ a prime ideal of R with $\dim R/\mathfrak{a} = 1$. In view of 2.6, it suffices to show that $\operatorname{Hom}_R(M, H^q_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-cofinite for all finitely generated R-modules M with $\operatorname{pd} M < \infty$. We prove this claim by induction on $\operatorname{pd} M = t$. The case $t = 0$, is clear by 1.2. Now assume that $t > 0$ and consider the exact sequence

$$0 \longrightarrow K \longrightarrow R^n \longrightarrow M \longrightarrow 0.$$

It follows that $\operatorname{pd} K \leq t - 1$. This short exact sequence yields the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, H^q_{\mathfrak{a}}(N)) \longrightarrow \operatorname{Hom}_R(R^n, H^q_{\mathfrak{a}}(N)) \xrightarrow{f} \operatorname{Hom}_R(K, H^q_{\mathfrak{a}}(N)).$$

Since, by [**4**, Theorem 2] the subcategory of $\mathfrak{a}$-cofinite R-modules is abelian, it follows that $\ker f \cong \operatorname{Hom}_R(M, H^q_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-cofinite. Therefore the claim follows by induction. ■

Theorem 2.8. *Let $\mathfrak{a}$ denote a principal ideal of the ring R. Let M and N be two finitely generated R-modules with $\operatorname{pd} M < \infty$. Then $H^p_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $p \geq 0$.*

Proof. By [**9**, Theorem 11.38], there is a Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M,N)$$

This implies the following exact sequence in view of [**11**, Ex. 5.2.2]. Note that $E_2^{p,q} = 0$ for $q \neq 0, 1$.

$$\longrightarrow E_2^{p,0} \xrightarrow{f} H^p_{\mathfrak{a}}(M,N) \xrightarrow{d} E_2^{p-1,1} \longrightarrow E_2^{p+1,0} \xrightarrow{g} H^{p+1}_{\mathfrak{a}}(M,N) \longrightarrow \dots .$$

Now, $\operatorname{im} f$ is a quotient of $E_2^{p,0}$ and so is finitely generated. Hence $\operatorname{im} f$ is $\mathfrak{a}$-cofinite, by 2.3 and 2.4(i). Also, $\ker g$ is $\mathfrak{a}$-cofinite by the same reason. By considering the short exact sequence

$$0 \longrightarrow \operatorname{im} d \longrightarrow E_2^{p-1,1} \longrightarrow \ker g \longrightarrow 0,$$

we deduce that $\operatorname{im} d$ is $\mathfrak{a}$-cofinite. Note that $E_2^{p-1,1}$ is $\mathfrak{a}$-cofinite by 2.7(i). Now from the short exact sequence

$$0 \longrightarrow \operatorname{im} f \longrightarrow H^p_{\mathfrak{a}}(M,N) \longrightarrow \operatorname{im} d \longrightarrow 0,$$

we deduce that $H^p_{\mathfrak{a}}(M,N)$ is a-cofinite for all $p \geq 0$. ■

Theorem 2.9. *Let $\mathfrak{p}$ denote a prime ideal of the complete local ring $(R, \mathfrak{m})$ with $\dim R/\mathfrak{p} = 1$, and M, N two finitely generated R-modules with $\operatorname{pd} M < \infty$. Then $H^i_{\mathfrak{p}}(M,N)$ is $\mathfrak{p}$-cofinite for all $i \geq 0$.*

Proof. There is a spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(M, H^q_{\mathfrak{p}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{p}}(M,N) = E^n.$$

It follows from 2.7(ii) that $E_2^{p,q}$ is $\mathfrak{p}$-cofinite for all p, q. By considering the sequence

$$\dots \longrightarrow E_2^{p-2,q+1} \xrightarrow{d_2^{p-2,q+1}} E_2^{p,q} \xrightarrow{d_2^{p,q}} E_2^{p+2,q-1} \longrightarrow \dots,$$

we deduce that $\operatorname{im} d_2^{p-2,q+1}$ and $\ker d_2^{p,q}$ are $\mathfrak{p}$-cofinite, by [**4**, Theorem 2]. Hence $E_3^{p,q} = \ker d_2^{p,q} / \operatorname{im} d_2^{p-2,q+1}$ is $\mathfrak{p}$-cofinite. By irritating this arguments we get that $E_r^{p,q} = \ker d_{r-1}^{p,q} / \operatorname{im} d_{r-1}^{p-r+1,q+r-2}$ is $\mathfrak{p}$-cofinite for all $r > 0$ and so $E_\infty^{p,q}$ is $\mathfrak{p}$-cofinite for all $p, q \geq 0$. There is a filtration

$$E^n = E^n_0 \supseteq \cdots \supseteq E^n_p \supseteq \cdots \supseteq E^n_n \supseteq E^n_{n+1} = 0,$$

such that $E^n_p/E^n_{p+1} \cong E^{p,n-p}_\infty$. Thus E^n_n is $\mathfrak{p}$-cofinite. Now, by applying 2.4(ii) repeatedly on the short exact sequences

$$0 \longrightarrow E^n_{p+1} \longrightarrow E^n_p \longrightarrow E^{p,n-p}_\infty \longrightarrow 0, p = 0, 1 \dots, n-1,$$

we deduce that E^n is $\mathfrak{p}$-cofinite, as required. ■

Many results concerning local cohomology in positive prime characteristic can be extend to generalized local cohomology. In particular the main results of [**7**] are also hold for generalized local cohomology.

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$, and $\mathfrak{a}$ an ideal of R. Let $\mathfrak{p}$ be a prime ideal of R and M a finitely generated R-module. Then*

(i) $\mu^i(\mathfrak{p}, H^j_{\mathfrak{a}}(M, R)) \leq \mu^i(\mathfrak{p}, \mathrm{Ext}^j_R(M/\mathfrak{a}M, R))$ *for all* $j \geq 0$. *In particular* $\mu^i(\mathfrak{p}, H^j_{\mathfrak{a}}(M, R))$ *is finite for all* $j \geq 0$ *and all* $i \geq 0$.

(ii) $\mathrm{Ass}_R(H^j_{\mathfrak{a}}(M, R)) \subseteq \mathrm{Ass}_R(\mathrm{Ext}^j_R(M/\mathfrak{a}M, R))$ *and so* $\mathrm{Ass}_R(H^j_{\mathfrak{a}}(M, R))$ *is finite for all* $j \geq 0$.

Proof. The proof is a straightforward adoption of the proof of [**7**, Theorem 2.1 and Corollary 2.3]. ■

Acknowledgments. We thank the referee whose comments make the paper more readable. Also, the first named author would like to thank the Institute for Studies in Theoretical Physics and Mathematics (Tehran) for financial support.

References

[1] J. Asadollahi, K. Khashyarmanesh and Sh. Salarian, *On the finiteness properties of the generalized local cohomology modules*, Comm. Algebra **30**(2), (2002), 859–867.

[2] M. H. Bijan-Zadeh, *A common generalization of local cohomology theories*, Glasgow Math. J. **21** (1980), 173-181.

[3] M. P. Brodmann, R. Y. Sharp: *'Local cohomology-An algebraic introduction with geometric applications'*, Cambr. Univ. Press, 1998.

[4] D. Delfino and T. Marley, *Cofinite modules and local cohomology*, J. Pure. Appl. Algebra **121** (1997), 45-52.

[5] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145-164.

[6] J. Herzog, *Komplex Auflösungen und Dualität in der lokalen algebra*, preprint, Universitüt Regensburg, 1974.

[7] C. Hunke and R. Y. Sharp, *Bass numbers of local cohomology modules* , Trans. AMS **339** (2), (1993), 765-779.

[8] K. I. Kawasaki, *Cofiniteness of local cohomology modules for principle ideals*, Bull. London. Math. Soc. **30** (1998), 241-246.

[9] J. Rotman, *Introduction to homological algebra*, Academic Press, 1979.

[10] N. Suzuki, *On the generalized local cohomology and its duality*, J. Math. Kyoto. Univ. **18** (1) (1978), 71-85.

[11] C. A. Weibel, *An introduction to homological algebra*, Camb.Univ. Press, 1994.

[12] S. Yassemi, *Generalized section functors*, J. Pure. Apple. Algebra **95** (1994), 103-119.

[13] S. Yassemi, *Cofinite modules* , Comm. Algebra **29**(6), (2001), 2333–2340.

[14] S. Yassemi, L. Khatami and T. Sharif, *Associated primes of generalized local cohomology modules*, Comm. Algebra **30**(1), (2002), 327-330.

[15] K. I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J. **147** (1997), 179-191.

K. Divaani-Aazar, Department of Mathematics, Az-Zahra University, Vanak, Post Code 19834, Tehran, Iran and Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran.

E-mail address: `kdivaani@ipm.ir`

R. Sazeedeh, Department of Mathematics, Uromeiyeh University, Uromeiyeh, Iran and Institute of Mathematics, University for Teacher Education, 599 Taleghani Avenue, Tehran 15614, Iran.